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Blow-ups and resolutions of strong Kähler with torsion metrics ☆

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Abstract

On a compact complex manifold we study the behaviour of strong Kähler with torsion (strong KT) structures under small deformations of the complex structure and the problem of extension of a strong KT metric. In this context we obtain the analogous result of Miyaoka extension theorem. Studying the blow-up of a strong KT manifold at a point or along a complex submanifold, we prove that a complex orbifold endowed with a strong KT metric admits a strong KT resolution. In this way we obtain new examples of compact simply-connected strong KT manifolds.

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1. Introduction

Let (M, J, g) be a Hermitian manifold of complex dimension n. By [19] there is a 1-parameter family of canonical Hermitian connections on M which can be distinguished by properties of their torsion tensor T. In particular, there exists a unique connection ∇^B satisfying $\nabla^B g = 0$, $\nabla^B J = 0$ for which $g(X, T(Y, Z))$ is totally skew-symmetric. The resulting 3-form can then

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be identified with $J\,dF$, where $F(\cdot,\cdot) = g(J\,\cdot\,,\cdot)$ is the fundamental 2-form associated to the Hermitian structure (J, g). This connection was used by Bismut in [7] to prove a local index formula for the Dolbeault operator when the manifold is non-Kähler. The properties of such a connection are related to what is called "*Kähler with torsion geometry*" and if $J\,dF$ is closed, or equivalently if F is $\partial\bar{\partial}$-closed, then the Hermitian structure (J, g) is strong KT and g is called a *strong KT* or a *pluriclosed* metric. The strong KT metrics have also applications in type II string theory and in 2-dimensional supersymmetric σ-models [17,35] and have relations with generalized Kähler structures (see for instance [3,16,20,24]).

The condition $\partial\bar{\partial}F = 0$ is obviously satisfied if $dF = 0$, i.e. if g is a Kähler metric. The interesting strong KT metrics for us are those ones which are not Kähler and therefore it is natural to investigate which properties that hold for Kähler manifolds can be generalized in the context of strong KT geometry.

In view of this, in the present paper, we study in particular the behaviour of strong KT structures under small deformations of the complex structure, the blow-up of a strong KT manifold at a point or along a complex submanifold and the problem of extension of a strong KT metric on a complex manifold.

The theory about strong KT manifolds in complex dimension at least three is completely different from that one on complex surfaces. Indeed, on a complex surface a Hermitian metric satisfying the strong KT condition is "*standard*" in the terminology of Gauduchon [18] and there exists a standard metric in the conformal class of any given Hermitian metric on a compact manifold. Therefore on a complex surface the strong KT condition is stable under small deformations of the complex structure.

Examples of compact strong KT manifolds of complex dimension three are given by nilmanifolds, i.e. compact quotients of nilpotent Lie groups by uniform discrete subgroups (see [15]). It is well known that these manifolds are not formal in the sense of [36] and cannot admit any Kähler metric unless they are tori (see [5,10,22]). More precisely, in [15] it was showed that if a nilmanifold of real dimension 6 admits a strong KT structure then the nilpotent Lie group has to be 2-step and therefore the nilmanifold has to be the total space of a torus bundle over a torus.

One of the examples found in [15] is the *Iwasawa manifold*, which can also be viewed as the total space of a $\mathbb{T}^2$-bundle over the torus $\mathbb{T}^4$. In contrast with the Kodaira–Spencer stability theorem [28] and the case of complex surfaces, in Section 2 we prove that on this manifold the condition strong KT is not stable under small deformations of the complex structure.

As in the Kähler case, in Section 3 we prove that the blow-up of a strong KT manifold at a point is still strong KT and more in general

Proposition 3.2. *Let M be a complex manifold endowed with a strong KT metric g. Let $Y \subset M$ be a compact complex submanifold. Then the blow-up $\tilde{M}_Y$ of M along Y has a strong KT metric.*

This result allows, starting from a strong KT manifold, to construct a new one.

The Kähler condition for a Hermitian metric can be characterized in terms of positive currents. Indeed, Harvey and Lawson in [21] proved that a compact complex manifold admits a Kähler metric if and only if there is no non-zero positive $(1,1)$-current which is the $(1,1)$-component of a boundary. By [12] also the strong KT condition can be studied in terms of positive currents. A result of Miyaoka [30] asserts that, if a (compact) complex manifold M has a Kähler metric in the complement of a point, then M is itself Kähler. By using the extension result of [1] about positive or negative plurisubharmonic currents, in Section 4 we prove the analogous result of the

Miyaoka's one for strong KT manifolds of complex dimension n with $n \geqslant 2$. As an application by using the previous extension theorem we show the following

Theorem 4.3. *Let M be a complex manifold of complex dimension $n \geqslant 2$. If $M \setminus \{p\}$ admits a strong KT metric, then there exists a strong KT metric on M.*

There are few examples of compact simply-connected strong KT manifolds; as far as we know, they are given by real compact semisimple Lie groups of even dimension [34]. Therefore it is interesting to investigate for new compact simply-connected strong KT manifolds. A natural way to obtain these is to consider resolutions of orbifolds and the typical example of orbifold is given by the quotient of a manifold by an action of a finite group with non-identity fixed point sets of codimension at least two (see [32]). In Section 5 we study resolutions of strong KT orbifolds. By using the result by Hironaka [23] that a complex orbifold admits a resolution, which is a obtained by a finite sequence of blow-ups and the results obtained about blow-ups, we prove the following

Theorem 5.4. *Let (M, J) be a complex orbifold of complex dimension n endowed with a J-Hermitian strong KT metric g. Then there exists a strong KT resolution.*

In the last two sections we apply this result to complex orbifolds constructed considering an action of a finite group on a torus, on a product of a Kodaira–Thurston surface with the 4-dimensional torus $\mathbb{T}^4$ and on a product of two Kodaira–Thurston surfaces. The strong KT resolutions that we get have a simpler topology with respect to the one of the three 8-dimensional nilmanifolds. Indeed, in the case of the torus we are able to construct a new compact simply-connected strong KT manifold and in the other two cases the strong KT resolutions have first Betti number equal to one. Resolutions for quotients of tori have been already considered by Joyce in order to obtain simply-connected compact manifolds with exceptional holonomy G_2 and *Spin*(7) in [26]. Moreover, in [13] and [9] orbifolds constructed starting with nilmanifolds have been recently used in order to get simply-connected compact non-formal symplectic manifolds.

2. Small deformations of strong KT metrics

We will recall some basic definitions and fix some notation. Let (M, J) be a complex manifold of complex dimension n and decompose as usual $d = \partial + \bar{\partial}$. Let g be a Hermitian metric on (M, J). The *fundamental* 2*-form* F is then defined by

$$F(X, Y) = g(JX, Y)$$

and has type $(1, 1)$ relative to the complex structure J.

Definition 2.1. The Hermitian metric g on (M, g) is said to be *strong Kähler with torsion*, or shortly, *strong KT*, if

$$\partial\bar{\partial}F = 0. \tag{1}$$

Clearly, condition (1) is weaker than the Kähler one and the previous definition includes for us the Kähler metrics.

In this section we will explore the stability of strong KT metrics under infinitesimal deformations of the complex structures.

By a well-known result by Kodaira–Spencer [28] the Kähler condition for a Hermitian metric is stable under small deformations of the complex structure underlying the Kähler structure. We will show that this does not hold for strong KT structures in real dimension higher than or equal to six. In real dimension 4 a strong KT metric is standard in the terminology of [18] and therefore the result holds.

Indeed, we recall that a Hermitian structure (J, g) on a manifold M of real dimension $2n$ is called *standard* in the terminology of [18] if

$$\partial\bar{\partial}F^{n-1} = 0,$$

or equivalently if the Lee form $\theta = -J * d * F$ is co-closed, where

$$F^{n-1} = \underbrace{F \wedge \ldots \wedge F}_{(n-1)\text{-times}}.$$

In particular, if $\theta = 0$ the Hermitian structure is said to be *balanced*.

By [18] for a compact complex manifold a standard metric can be found in the conformal class of any given Hermitian metric. Since on a complex surface a strong KT metric is standard, a small deformation of the complex structure on a complex surface preserves the strong KT condition. In higher dimensions the strong KT condition is not anymore equivalent to the standard one, in fact by [2] a strong KT metric is standard only if

$$|dF|^2 = (n-1)|\theta \wedge F|^2,$$

where θ is the Lee form of the Hermitian structure (J, g) and by $|\cdot|$ we denote the norm of the form. Therefore, if $n > 2$ a strong KT metric is not necessarily standard. An example of compact strong KT manifold of complex dimension three is given by the Iwasawa manifold $\mathbb{I}(3)$, which is the compact quotient of the complex Heisenberg group

$$H_3^{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z_j \in \mathbb{C},\ j = 1, 2, 3 \right\}$$

by the uniform discrete subgroup Γ for which z_j are Gaussian integers. By [15, Theorem 1.2] for this manifold and more in general for any nilmanifold of complex dimension three the strong KT condition depends only on the underlying complex structure. This allows us in contrast with the case $n = 2$ to prove the following

Theorem 2.2. *On the Iwasawa manifold $\mathbb{I}(3) = \Gamma\backslash H_3^{\mathbb{C}}$ the condition for a Hermitian metric to be strong KT is not stable under small deformations of the complex structure underlying the strong KT structure.*

In order to prove the theorem we will construct an explicit deformation of a complex structure underlying a strong KT structure that does not remain strong KT.

Let $\mathfrak{n}_{t,s}$ be the family of 2-step nilpotent Lie algebras with structure equations

$$\begin{cases} de^i = 0, \quad i = 1, \ldots, 4, \\ de^5 = t(e^1 \wedge e^2 + 2e^3 \wedge e^4) + s(e^1 \wedge e^3 - e^2 \wedge e^4), \\ de^6 = s(e^1 \wedge e^4 + e^2 \wedge e^3), \end{cases}$$

with t and s real numbers and $s \neq 0$. This family was already considered in [14] for Hermitian structures whose Bismut connection has holonomy in $SU(3)$ and it was proved that for any t and $s \neq 0$ the Lie algebra $\mathfrak{n}_{t,s}$ is isomorphic to the Lie algebra of the complex Heisenberg group $H_3^{\mathbb{C}}$ with structure equations

$$\begin{cases} de^i = 0, \quad i = 1, \ldots, 4, \\ de^5 = e^1 \wedge e^3 - e^2 \wedge e^4, \\ de^6 = e^1 \wedge e^4 + e^2 \wedge e^3 \end{cases}$$

(compare also [29, Examples 6.1 and 6.5]).

Take the almost complex structure J on $\mathfrak{n}_{t,s}$ given by

$$Je^1 = e^2, \qquad Je^3 = e^4, \qquad Je^5 = Je^6. \tag{2}$$

For the associated $(1,0)$-forms

$$\varphi^1 = e^1 + ie^2, \qquad \varphi^2 = e^3 + ie^4, \qquad \varphi^3 = e^5 + ie^6,$$

we have that

$$d\varphi^i = 0, \quad i = 1, 2,$$
$$d\varphi^3 = -\frac{1}{2}it\big(\varphi^1 \wedge \overline{\varphi}^1 + 2\varphi^2 \wedge \overline{\varphi}^2\big) + s\varphi^1 \wedge \varphi^2,$$

and therefore J is integrable.

In this way the Iwasawa manifold $\mathbb{I}(3) = \Gamma\backslash H_3^{\mathbb{C}}$ is endowed with a family of complex structures $J_{t,s}$, with $t, s \in \mathbb{R}$ and $s \neq 0$.

Note that for $t = 0$ and $s = 1$ the complex structure J coincides with the bi-invariant complex structure J_0 on the complex Heisenberg group. The complex structure J_0 cannot admit any compatible strong KT metric, since otherwise it has to be balanced and by [15] the balanced condition is complementary to the strong KT one.

We will show that the Iwasawa manifold $(\mathbb{I}(3), J_{t,s})$ admits a strong KT metric compatible with $J_{t,s}$ if and only if $t^2 = s^2$. Indeed, by [15, Lemma 1.3], if g is a left-invariant Riemannian metric compatible with $J_{t,s}$, g is strong KT if and only if

$$\partial\overline{\partial}\big(\varphi^3 \wedge \overline{\varphi}^3\big) = 0 = \big(t^2 - s^2\big)\varphi^1 \wedge \varphi^2 \wedge \overline{\varphi}^1 \wedge \overline{\varphi}^2.$$

By [37, Proposition 3.1] and [14] if there exists a non-left-invariant strong KT metric compatible with $J_{t,s}$, then there is also a left-invariant one. Therefore this is only possible if $t^2 = s^2$.

Thus if $t = s = 1$ the Iwasawa manifold has a strong KT metric g compatible with $J_{1,1}$, but for any $t \neq s \neq 1$ there exists no a strong KT metric compatible with the complex structure $J_{t,s}$.

In a similar way one can show that generically on a nilmanifold of complex dimension three the condition strong KT is not stable under small deformations of the underlying complex structure, but one can also construct a family of strong KT structures. For instance consider the family of Lie algebras $\mathfrak{n}_{t,s}$ with structure equations

$$\begin{cases} de^i = 0, \quad i = 1, \dots, 5, \\ de^6 = t^2 e^1 \wedge e^2 + ts(e^1 \wedge e^4 - e^2 \wedge e^3) + s^2 e^3 \wedge e^4. \end{cases}$$

For any real numbers t, s such that $t^2 + s^2 \neq 0$ the Lie algebra $\mathfrak{n}_{t,s}$ is isomorphic to the Lie algebra of $H_3 \times \mathbb{R}^3$, where H_3 is the real 3-dimensional Heisenberg Lie group. Moreover, for any t, s the complex structure J defined by (2) gives rise to a strong KT structure.

3. Blow-up of strong KT manifolds

We start by proving that the blow-up of a strong KT manifold at a point is still strong KT, as in the Kähler case (see for example [6]).

Proposition 3.1. *Let (M, J, g) be a strong KT manifold of complex dimension n and $\tilde{M}_p$ be the blow-up of M at a point $p \in M$. Then $\tilde{M}_p$ admits a strong KT structure.*

Proof. Let $z = (z_1, \dots, z_n)$ be holomorphic coordinates in an open set U centered around the point $p \in M$. We recall that the blow-up $\tilde{M}_p$ of M is the complex manifold obtained by adjoining to $M \setminus \{p\}$ the manifold

$$\tilde{U} = \bigl\{(z, l) \in U \times \mathbb{CP}^{n-1} \bigm| z \in l\bigr\}$$

by using the isomorphism

$$\tilde{U} \setminus \{z = 0\} \cong U \setminus \{p\}$$

given by the projection $(z, l) \to z$. In this way there is a natural projection $\pi : \tilde{M}_p \to M$ extending the identity on $M \setminus \{p\}$ and the exceptional divisor $\pi^{-1}(p)$ of the blow-up is naturally isomorphic to the complex projective space $\mathbb{CP}^{n-1}$.

If we denote by F the fundamental 2-form associated with the strong KT metric g, then the 2-form $\pi^* F$ is $\partial\bar{\partial}$-closed since π is holomorphic, but it is not positive definite on $\pi^{-1}(M \setminus \{p\})$. As in the Kähler case, let h be a $\mathcal{C}^\infty$-function having support in U, i.e. $0 \leqslant h \leqslant 1$ and $h = 1$ in a neighborhood of p. On $U \times (\mathbb{C}^n \setminus \{0\})$ consider the 2-form

$$\gamma = i\partial\bar{\partial}\bigl(\bigl(p_1^* h\bigr) p_2^* \log \|\cdot\|^2\bigr),$$

where p_1 and p_2 denote the two projections of $U \times (\mathbb{C}^n \setminus \{0\})$ on U, $\mathbb{C}^n \setminus \{0\}$ respectively.

Let ψ be the restriction of γ to $\tilde{M}_p$. Then there exists a small enough real number ϵ such that the 2-form $\tilde{F} = \epsilon\psi + \pi^* F$ is positive definite. Since $\tilde{F}$ is $\partial\bar{\partial}$-closed, it defines a strong KT metric on the blow-up $\tilde{M}_p$. $\square$

As a consequence of Proposition 3.1, it is possible to construct new examples of strong KT manifolds by blowing-up a given strong KT manifold M at one or more points. Moreover, the homology groups of the two manifolds M and $\tilde{M}_p$ are related by

$$H_i(\tilde{M}_p) = H_i(M) \oplus H_i\big(\mathbb{C}P^{n-1}\big), \quad i \geqslant 1.$$

Note that in view of Theorem 4.7, the blow-up of the strong KT nilmanifolds given in [15] cannot admit any Kähler structure.

Proposition 3.1 can be generalized to the blow-up of a strong KT manifold along a compact complex submanifold. Indeed

Proposition 3.2. *Let M be a complex manifold endowed with a strong KT metric g. Let $Y \subset M$ be a compact complex submanifold. Then the blow-up $\tilde{M}_Y$ of M along Y has a strong KT metric.*

Proof. Let $\pi : \tilde{M}_Y \to M$ be the holomorphic projection. By construction $\pi : \tilde{M} \setminus \pi^{-1}(Y) \to M \setminus Y$ is a biholomorphism and $\pi^{-1}(Y) \cong \mathbb{P}(\mathcal{N}_{Y|M})$, where $\mathbb{P}(\mathcal{N}_{Y|M})$ is the projectified of the normal bundle of Y. Let F be the fundamental 2-form of the strong KT metric on M. There exists a holomorphic line bundle L on $\tilde{M}_Y$ such that L is trivial on $\tilde{M}_Y \setminus \pi^{-1}(Y)$ and such that its restriction to $\pi^{-1}(Y)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y|M})}(1)$.

Let h be a Hermitian structure on $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Y|M})}(1)$ and ω be the corresponding Chern form. Let $\{U_i\}_{i\in I}$ be an open covering of M which trivializes the line bundle L. By using a partition of unity subordinate to $\{U_i\}_{i\in I}$, it follows that the metric h can be extended to a metric structure $\hat{h}$ on L, in such a way that $\hat{h}$ is the flat metric structure on the complement of a compact neighborhood W of Y induced by the trivialization of L on $\tilde{M}_Y \setminus \pi^{-1}(Y)$. Therefore, the Chern curvature $\hat{\omega}$ of L vanishes on $M \setminus W$ and $\hat{\omega}|_{\mathbb{P}(\mathcal{N}_{Y|M})} = \omega$.

Hence, since Y is compact, there exists $\epsilon \in \mathbb{R}$, $\epsilon > 0$, small enough, such that

$$\tilde{F} = \pi^* F + \epsilon\hat{\omega}$$

is positive definite. Moreover, $\partial\bar{\partial}\tilde{F} = 0$, so that $\tilde{F}$ gives rise to a strong KT metric on $\tilde{M}_Y$. □

By [8] the cohomology groups of the two manifolds M and $\tilde{M}_Y$ are related by

$$H^*(\tilde{M}_Y) = \pi^* H^*(M) \oplus H^*\big(\mathbb{P}(\mathcal{N}_{Y|M})\big)/\pi^* H^*(Y)$$

and therefore for the corresponding Poincaré polynomials we have

$$P_{\tilde{M}_Y}(t) = P_M(t) + P_Y(t)\left(\sum_{j=1}^{n-k-1} t^{2j}\right),$$

where $k = \dim_{\mathbb{C}} Y$. In particular, for the first Betti number we get

$$b_1(\tilde{M}_Y) = b_1(M).$$

4. Extension of strong KT metrics

We start by fixing some notation and recalling some known facts on positive currents. For our purposes it is enough to consider an open set $\Omega \subset \mathbb{C}^n$.

Denote by $\Lambda^{p,q}(\Omega)$ (respectively by $\mathcal{D}^{p,q}(\Omega)$) the space of (p,q)-forms (respectively (p,q)-forms with compact support) on Ω. On $\mathcal{D}^{p,q}(\Omega)$ consider the $\mathcal{C}^\infty$-topology. By definition, the *space of currents* of *bi-dimension* (p,q) or of *bi-degree* $(n-p,n-q)$ is the topological dual $\mathcal{D}'_{p,q}(\Omega)$ of $\mathcal{D}^{p,q}(\Omega)$. A current of bi-dimension (p,q) on Ω can be identified with a $(n-p, n-q)$-form on Ω with coefficients distributions. The *support* of a current $T \in \mathcal{D}'_{p,q}(\Omega)$, denoted by $\mathrm{supp}(T)$, is the smallest closed set C such that the restriction of T to $\mathcal{D}^{p,q}(\Omega \setminus C)$ is zero.

A current $T \in \mathcal{D}'_{p,q}(\Omega)$ is said to be of *order* 0 if its coefficients are measures and is said to be *normal* if T and dT are currents of order 0.

A current T of bi-dimension (p,p) is said to be *real* if $T(\varphi) = T(\overline{\varphi})$, for any $\varphi \in \mathcal{D}^{p,p}(\Omega)$. Therefore, if $T \in \mathcal{D}'_{p,p}(\Omega)$ is real, then we may write

$$T = \sigma_{n-p} \sum_{I,\bar{J}} T_{I\bar{J}}\, dz_I \wedge d\bar{z}_J,$$

where $\sigma_{n-p} = \frac{i^{(n-p)^2}}{2^{(n-p)}}$, $T_{I\bar{J}}$ are distributions on Ω such that $T_{J\bar{I}} = \overline{T}_{I\bar{J}}$ and I, J are multi-indices of length $n-p$, $I = (i_1, \ldots, i_{n-p})$, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_{n-p}}$.

A real current $T \in \mathcal{D}'_{p,p}(\Omega)$ is *positive* if,

$$T\big(\sigma_p \varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p\big) \geqslant 0$$

for any choice of $\varphi^1, \ldots, \varphi^p \in \mathcal{D}^{1,0}(\Omega)$, where $\sigma_p = \frac{i^{p^2}}{2^p}$. A current T is said to be *strictly positive* if $\varphi^1 \wedge \ldots \wedge \varphi^p \neq 0$ implies $T(\sigma_p \varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p) > 0$.

Recall that if T is a positive current of bi-degree (p,p), then T is of order 0.

A real current T of bi-dimension (p,p) on Ω is said to be *negative* if the current $-T$ is positive and *plurisubharmonic* if $i\partial\overline{\partial}T$ is positive.

If F is the fundamental 2-form of a Hermitian structure on a complex manifold M, then F corresponds to a real strictly positive current of bi-degree $(1,1)$. In particular, if the Hermitian structure is strong KT, then the corresponding current is $\partial\overline{\partial}$-closed.

An important class of $\partial\overline{\partial}$-closed currents is given by the (p,p)-components of a boundary. We recall that a current T of bi-degree (p,p) is called the *(p,p)-component of a boundary* if there exists a real current S of bi-degree $(p,p-1)$ such that $T = \partial\overline{S} + \overline{\partial}S$. In [21] Harvey and Lawson proved that a compact complex manifold has a Kähler metric if and only if there is no non-zero positive current of bi-dimension $(1,1)$ which is the $(1,1)$-component of a boundary. By [12] this characterization of the Kähler condition can be generalized in the context of strong KT geometry showing that a compact complex manifold admits a strong KT metric if and only if there is no non-zero positive current of bi-dimension $(1,1)$ which is $\partial\overline{\partial}$-exact.

In [30] Miyaoka showed that if a complex manifold M has a Kähler metric in the complement of a point, then the manifold M itself is Kähler.

In order to prove a similar result for strong KT structures we need to recall the following extension theorem (see [1, Main Theorem 5.6]).

Theorem 4.1. *Let Y be an analytic subset of $\Omega \subset \mathbb{C}^n$. If T is a plurisubharmonic, negative current of bi-dimension (p,p) on the complement $\Omega \setminus Y$ of Y in Ω and* $\dim_{\mathbb{C}} Y < p$, *then there exists the simple (or trivial) extension T^0 of T across Y and T^0 is plurisubharmonic.*

If $T = \sigma_{n-p} \sum_{I,\bar{J}} T_{I\bar{J}}\, dz_I \wedge d\bar{z}_J$, on $\Omega \setminus Y$, with $T_{I\bar{J}}$ measures, then the current T^0 on Ω is defined by extending the $T_{I\bar{J}}$ to zero on Y.

Finally, we recall (see e.g. [11, Corollary 2.11, p. 181]) the following corollary of the Support theorem [11, Theorem 2.10, p. 180].

Theorem 4.2. *Let T be normal current of bi-dimension (p,p) on $\Omega \subset \mathbb{C}^n$. If* $\operatorname{supp}(T)$ *is contained in an analytic subset Y of Ω such that* $\dim_{\mathbb{C}} Y < p$, *then* $T = 0$.

By using the previous results we are ready to prove the following

Theorem 4.3. *Let M be a complex manifold of complex dimension $n \geqslant 2$. If $M \setminus \{p\}$ admits a strong KT metric, then there exists a strong KT metric on M.*

The proof of Theorem 4.3 is a consequence of the following

Proposition 4.4. *Let F be the fundamental 2-form of a strong KT metric on $\mathbb{B}^n(r) \setminus \{0\}$, $n \geqslant 2$. Then there exist $0 < R \leqslant r$ and $\hat{F} \in \Lambda^{1,1}(\mathbb{B}^n(R))$ such that*

(i) *$\hat{F}$ is the fundamental 2-form of a strong KT metric on $\mathbb{B}^n(R)$,*
(ii) *$\hat{F} = F$ on $\mathbb{B}^n(R) \setminus \mathbb{B}^n(\frac{2}{3}R)$.*

Proof. A key tool in the proof of the proposition is the following result by [4, Theorem 1.15] on $\partial\bar{\partial}$-closed currents, which is based on an argument given by Siu [33, p. 121], for d-closed l-currents with measure coefficients.

Theorem 4.5. *Let T be a current of bi-degree (h,k) on Ω. If T is of order* 0 *and $i\partial\bar{\partial}T = 0$, then, locally,*

$$T = \partial G + \bar{\partial} H,$$

for suitable currents G and H with locally integrable functions as coefficients.

For the sake of completeness, we will give the proof of Theorem 4.5 (see [4, Theorem 1.15] and [33, p. 121]).

Proof of Theorem 4.5. Consider

$$\Lambda^{1,0}(\Omega) \oplus \Lambda^{0,1}(\Omega) \xrightarrow{\partial+\bar{\partial}} \Lambda^{1,1}(\Omega) \xrightarrow{i\partial\bar{\partial}} \Lambda^{2,2}(\Omega).$$

Then, according to the theorem of Hodge for elliptic complexes (see e.g. [38, p. 235]), the differential operator

$$\square = (\partial + \bar{\partial})(\partial + \bar{\partial})^*(\partial + \bar{\partial})(\partial + \bar{\partial})^* + (i\partial\bar{\partial})^*(i\partial\bar{\partial})$$

is elliptic. In view of [25, Theorem 7.1.20], there exists a fundamental solution E of the differential operator $\Box$ given by

$$E = E_0 - Q(z)\log|z|,$$

where E_0 is a matrix of homogeneous distributions of degree $4-2n$, smooth in $\mathbb{C}^n \setminus \{0\}$, Q is a matrix of polynomials which vanishes identically for $2n > 4$ and it is constant for $2n = 4$.

Set

$$L = (\partial + \bar{\partial})^*(\partial + \bar{\partial})(\partial + \bar{\partial})^*$$

and let λ be a $\mathcal{C}^\infty$-function, with compact support contained in Ω and such that $\lambda(z) = 1$ on a ball $U' \subset \Omega$, $0 \in U'$. Then, we have

$$\begin{aligned}
\lambda T &= \Box(E * \lambda T) \\
&= (\partial + \bar{\partial})\big(L(E * \lambda T)\big) + \bar{\partial}^*\partial^* E * \partial\bar{\partial}(\lambda T) \\
&= (\partial + \bar{\partial})\big(L(E * \lambda T)\big) + \bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T + \lambda\partial\bar{\partial}T) \\
&= (\partial + \bar{\partial})\big(L(E * \lambda T)\big) + \bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T).
\end{aligned}$$

By a direct computation, it turns out that the current $\bar{\partial}^*\partial^* E$ has locally integrable functions as coefficients. Hence, the coefficients of the current

$$\bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T)$$

are locally integrable functions, since they are obtained as convolutions of locally integrable functions with measures.

Since $\lambda = 1$ on U', $\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T$ vanishes identically on U'. Therefore, by [25, Theorem 4.2.5], it follows that

$$\operatorname{sing\,supp}\big(\bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T)\big) \subset \Omega \setminus U'$$

where sing supp denotes the singular support of a current, i.e. the complement in Ω of the open set A such that the restriction of the current to A is smooth. Hence,

$$\bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T)$$

is $\mathcal{C}^\infty$ on U'. Furthermore, it is $\partial\bar{\partial}$-closed. Consequently, there exist a $(h-1,k)$-form ϕ and a $(h,k-1)$-form ψ on U', such that

$$\bar{\partial}^*\partial^* E * (\partial\bar{\partial}\lambda \wedge T - \bar{\partial}\lambda \wedge \partial T + \partial\lambda \wedge \bar{\partial}T) = \partial\phi + \bar{\partial}\psi.$$

Therefore, on U', we can write

$$T = \partial G + \bar{\partial} H$$

where

$$G = L(E * \lambda T) + \phi, \qquad H = L(E * \lambda T) + \psi. \qquad \Box \tag{3}$$

Now, let us start with the proof of Proposition 4.4.

By hypothesis, F is a positive $\partial\bar{\partial}$-closed $(1,1)$-form on $\mathbb{B}^n(r)\setminus\{0\}$. Let $T=-F$; then T is a real (strictly) negative $\partial\bar{\partial}$-closed current of bi-degree $(1,1)$ on $\mathbb{B}^n(r)\setminus\{0\}$. In view of the result in [1, Main Theorem 5.6] (see Theorem 4.1 above), applied to the case $Y=\{0\}$, the simple extension T^0 of T on the ball $\mathbb{B}^n(r)$, defined by

$$T^0(\varphi)=\int\limits_{\mathbb{B}^n(r)\setminus\{0\}} F\wedge\varphi,\quad \forall\varphi\in\mathcal{D}^{n-1,n-1}\big(\mathbb{B}^n(r)\big),$$

is negative on $\mathbb{B}^n(r)$.

Consider now the current $i\partial\bar{\partial}T^0$. We have that $i\partial\bar{\partial}T^0$ is positive and consequently it is of order 0. Moreover, $d(i\partial\bar{\partial}T^0)=0$. Therefore, $i\partial\bar{\partial}T^0$ is a normal current of bi-degree $(2,2)$ on the ball $\mathbb{B}^n(r)$. Hence, by the corollary of the Support theorem (see Theorem 4.2 above) we obtain that

$$i\partial\bar{\partial}T^0=0\quad\text{on }\mathbb{B}^n(r).$$

Therefore, T^0 is a negative $\partial\bar{\partial}$-closed current of bi-degree $(1,1)$ on the ball $\mathbb{B}^n(r)$. Moreover, the coefficients of T^0 are measures. Observe that T^0 is smooth on $\mathbb{B}^n(r)\setminus\{0\}$.

Set $F^0=-T^0$. Then F^0 is clearly a real positive $\partial\bar{\partial}$-closed current of bi-degree $(1,1)$ on $\mathbb{B}^n(r)$ and it is strictly positive on $\mathbb{B}^n(r)\setminus\{0\}$.

In view of [4, Theorem 1.15] (see Theorem 4.5 above) and reality of F^0, we may write

$$F^0=\partial G+\bar{\partial}\overline{G}\quad\text{on }\mathbb{B}^n(R)$$

for some $0<R\leqslant r$, where G is a current of bi-degree $(0,1)$ whose coefficients are locally integrable functions on $\mathbb{B}^n(R)$.

As a consequence of (3), G is in fact smooth on $\mathbb{B}^n(R)\setminus\{0\}$. Indeed,

$$G=L(E*\lambda F_0)+\phi$$

and the fundamental solution E of $\square$ and the current F_0 are smooth on $\mathbb{B}^n(R)\setminus\{0\}$. Again by [25, Theorem 4.2.5], it follows that

$$\operatorname{sing\,supp}(E*\lambda F_0)\subset\operatorname{sing\,supp}E+\operatorname{sing\,supp}\lambda F_0.$$

Therefore, G is smooth on $\mathbb{B}^n(R)\setminus\{0\}$.

Now we are going to define a strong KT metric with fundamental 2-form $\hat{F}$ on $\mathbb{B}^n(R)$ as in the statement.

Set

$$G=\sum_{j=1}^n u_j\,d\bar{z}_j,$$

where u_i are locally integrable on $\mathbb{B}^n(R)$ and smooth on $\mathbb{B}^n(R)\setminus\{0\}$.

Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$-function on $\mathbb{C}^n$ such that

(a) ρ is radial and $\mathrm{supp}(\rho(z)) \subset \mathbb{B}^n(\frac{2}{3}R)$,
(b) $\rho(z) = 1$, $\forall z \in \overline{\mathbb{B}^n(\frac{1}{3}R)}$,
(c) $\int_{\mathbb{C}^n} \rho(z)\, dz = 1$.

Define

$$\tilde{u}_{j\epsilon}(z) = \int_{\mathbb{C}^n} u_j\big(z - \epsilon\rho(z)\zeta\big)\rho(\zeta)\, d\zeta, \quad j = 1, \ldots, n.$$

By using the conditions (a), (b) and (c) it can be checked that, for any $j = 1, \ldots, n$, $\tilde{u}_{j\epsilon}(z)$ is a $\mathcal{C}^\infty$-function on $\mathbb{B}^n(R)$ such that

$$\tilde{u}_{j\epsilon}(z) = u_j(z) \quad \text{on } \mathbb{B}^n(R) \Big\backslash \mathbb{B}^n\left(\frac{2}{3}R\right).$$

Now, if we set

$$G_\epsilon = \sum_{j=1}^n \tilde{u}_{j\epsilon}\, d\bar{z}_j,$$

then

$$\tilde{F}_\epsilon = \partial G_\epsilon + \bar{\partial}\overline{G}_\epsilon$$

is a real $\partial\bar{\partial}$-closed $(1,1)$-form on $\mathbb{B}^n(R)$ such that $\tilde{F}_\epsilon = F$ on $\mathbb{B}^n(R) \setminus \mathbb{B}^n(\frac{2}{3}R)$.

Note that, for ϵ small enough, $\tilde{F}_\epsilon$ is strictly positive on $\mathbb{B}^n(R) \setminus \{0\}$ and positive on $\mathbb{B}^n(R)$. Therefore, in order to get the strict positivity on the whole ball $\mathbb{B}^n(R)$ we need to perturb $\tilde{F}_\epsilon$. To such a purpose, let $h : \mathbb{C}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$-function on $\mathbb{C}^n$ such that

(i) $\mathrm{supp}(h(z)) \subset \mathbb{B}^n(\frac{1}{3}R)$,
(ii) $h(z) = 1$, $\forall z \in \overline{\mathbb{B}^n(\frac{1}{6}R)}$.

Then define

$$\hat{F}_\epsilon = \partial\left(G_\epsilon + \frac{i}{2}c\bar{\partial}\big(h(z)|z|^2\big)\right) + \bar{\partial}\overline{\left(G_\epsilon + \frac{i}{2}c\bar{\partial}\big(h(z)|z|^2\big)\right)},$$

where c is a real number. We immediately obtain

$$\hat{F}_\epsilon = \tilde{F}_\epsilon + ic\partial\bar{\partial}\big(h(z)|z|^2\big)$$

and consequently $\hat{F}_\epsilon = \tilde{F}_\epsilon$ on $\mathbb{B}^n(R) \setminus \mathbb{B}^n(\frac{1}{3}R)$. Finally, $\hat{F}_\epsilon$ is strictly positive on $\mathbb{B}^n(\frac{1}{6}R)$ and, by choosing the positive real number c small enough, we get that $\hat{F}_\epsilon$ is also strictly positive on $\overline{\mathbb{B}^n(\frac{1}{3}R)} \setminus \mathbb{B}^n(\frac{1}{6}R)$.

Therefore, $\hat{F}_\epsilon$ is a real, (strictly) positive, $\partial\bar{\partial}$-closed, $(1,1)$-form on $\mathbb{B}^n(R)$ and thus it gives rise to a strong KT metric we were looking for. □

Proof of Theorem 4.3. Let V be a disc around $p \in M$. Then by Proposition 4.4, there exist a smaller disc $U \subset V$ and a positive, $\partial\bar{\partial}$-closed $(1,1)$-form $\hat{F}$ on V such that $\hat{F} = F$ on $V \setminus U$. Therefore, the $(1,1)$-form defined by

$$\begin{cases} F_q & \text{if } q \in M \setminus U, \\ \hat{F}_q & \text{if } q \in V \end{cases}$$

is the fundamental 2-form of a strong KT metric on M. □

Remark 4.6. If $n = 1$, then any Hermitian metric is Kähler and, consequently, for $n = 1$, the proof of Theorem 4.3 follows at once.

As an application of Theorem 4.3 we can prove the following

Theorem 4.7. *Let M be a complex manifold of complex dimension $n \geqslant 2$ and $\tilde{M}$ be the blow-up of M at a point $p \in M$. Then $\tilde{M}$ has a strong KT metric if and only if M admits a strong KT metric.*

Proof. Assume that $\tilde{M}$ has a strong KT metric. Let $E = \pi^{-1}(p)$. Then $\pi : \tilde{M} \setminus E \to M \setminus \{p\}$ is a biholomorphism. Therefore $M \setminus \{p\}$ has a strong KT metric. By Theorem 4.3, M has a strong KT metric.

The other implication is given by Proposition 3.1. □

5. Strong KT orbifolds and resolutions

Orbifolds are a special class of singular manifolds and they have been used by Joyce in [26] to construct compact manifolds with special holonomy and in [13] to obtain non-formal symplectic compact manifolds.

We start by recalling the following (see e.g. [26])

Definition 5.1. A *complex orbifold* is a singular complex manifold M of dimension n such that each singularity p is locally isomorphic to U/G, where U is an open set of $\mathbb{C}^n$, G is a finite subgroup of $GL(n,\mathbb{C})$ acting linearly on U with the only one fixed point p. Moreover, the set S of singular points of M of the orbifold M has real codimension at least two.

A very easy method to construct complex orbifolds is to consider a holomorphic action of a finite group G on a manifold M, with non-identity fixed point sets of real codimension at least two. The quotient M/G is thus by definition a complex orbifold.

Since orbifolds have a mild form of singularities, many good properties for manifolds also hold for the orbifolds. For instance, the notions of smooth r-forms and (p,q)-forms make sense on complex orbifolds. The de Rham and Dolbeault cohomologies are well defined for orbifolds and they have many of the usual properties that they have in the case of complex manifolds.

More precisely, an *r-orbifold differential form* on a complex orbifold (M,J) is an r-differential form on M that is G-invariant in any chart U/G of M and a differential operator

$d : \Lambda^r_{\mathrm{orb}}(M) \to \Lambda^{r+1}_{\mathrm{orb}}(M)$ is defined on the complex $\Lambda_{\mathrm{orb}}(M)$ of orbifold differential forms on M. For the complex space $\Lambda^r_{\mathrm{orb}}(M) \otimes \mathbb{C}$ we have

$$\Lambda^r_{\mathrm{orb}}(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_{\mathrm{orb}}(M).$$

The elements of $\Lambda^{p,q}_{\mathrm{orb}}(M)$ are called *(p,q)-forms*, and, according to the above decomposition, the differential d splits as $d = \partial + \bar{\partial}$, as usual.

There is a natural notion of Hermitian metric on complex orbifolds. A *Hermitian metric g* on a complex orbifold (M, J) is a J-Hermitian metric in the usual sense on the non-singular part of (M, J) and G-invariant in any chart U/G. In such a case, for any chart U/G, we have $G \subset U(n)$.

Definition 5.2. A Hermitian metric g on a complex orbifold (M, J) is said to be *strong KT* if the fundamental 2-form F of g satisfies

$$\partial\bar{\partial}F = 0.$$

We recall that in general a resolution $(\tilde{M}, \pi)$ of a singular complex variety M is a normal, non-singular complex variety $\tilde{M}$ with a proper surjective birational morphism $\pi : \tilde{M} \to M$. We are interested in particular to resolve singularities of a complex orbifold endowed with a strong KT metric in order to obtain a smooth complex manifold admitting a strong KT metric.

Definition 5.3. Let (M, J, g) be a complex orbifold endowed with a strong KT metric g. A *strong KT resolution* of (M, J, g) is the datum of a smooth complex manifold $(\tilde{M}, \tilde{J})$ endowed with a $\tilde{J}$-Hermitian strong KT metric $\tilde{g}$ and of a map $\pi : \tilde{M} \to M$, such that

(i) $\pi : \tilde{M} \setminus E \to M \setminus S$ is a biholomorphism, where S is the singular set of M and $E = \pi^{-1}(S)$ is the *exceptional set*;
(ii) $\tilde{g} = \pi^* g$ on the complement of a neighborhood of E.

In view of the Hironaka Resolution of Singularities theorem [23], the singularities of any complex variety can be resolved by a finite number of blow-ups. Indeed, if M is a complex algebraic variety, then there exists a resolution $\pi : \tilde{M} \to M$, which is the result of a finite sequence of blow-ups of M. This means that there are varietes $M = M_0, M_1, \ldots, M_k = \tilde{M}$, such that M_j is a blow-up of M_{j-1} along some subvariety with projection $\pi_j : M_j \to M_{j-1}$ and the map $\pi : \tilde{M} \to M$ is given by the composition $\pi = \pi_1 \circ \cdots \circ \pi_k$.

Therefore applying Hironaka's theorem and the results about blow-ups obtained in Section 3 we can prove the following

Theorem 5.4. *Let (M, J) be a complex orbifold of complex dimension n endowed with a J-Hermitian strong KT metric g. Then there exists a strong KT resolution.*

Proof. Let $p \in S$ be a singular point of M. Take a chart $U_p = \mathbb{B}^n(r)/G_p$, where $\mathbb{B}^n(r) \subset \mathbb{C}^n$ is the ball of radius r in $\mathbb{C}^n$. Then $X = \mathbb{C}^n/G_p$ is an affine algebraic variety which has the origin as the only singular point. By Hironaka (see [23]), there exists a resolution $\pi_X : \tilde{X} \to X$ which is a quasi-projective variety and it is obtained by a finite sequence of blow-ups. The set $E = \pi_X^{-1}(0)$ is a complex submanifold of $\tilde{X}$. Set $\tilde{U} = \pi_X^{-1}(U_p)$. By identifying $\tilde{U} \setminus E$ with $U_p \setminus \{p\}$, define

$$\tilde{M} = \big(M \setminus \{p\}\big) \cup \tilde{U}.$$

Now, we define a strong KT metric on $\tilde{M}$ that coincides with $\pi^* g$ on the complement of a neighborhood of the exceptional set E.

Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be the function defined by $\rho(z) = \sum_{j=1}^n z_j \bar{z}_j$, $\omega_0 = i\partial\bar{\partial}\rho$ be the standard Kähler form in $\mathbb{C}^n$ and $\iota : \mathbb{B}^n(r) \hookrightarrow \mathbb{C}^n$. Let h be a non-negative real-valued $\mathcal{C}^\infty$-function such that

$$h \equiv 1 \quad \text{on } \mathbb{B}^n\left(\frac{1}{3}r\right)\Big/G_p,$$

$$h \equiv 0 \quad \text{on } \left(\mathbb{B}^n(r)\Big\backslash\mathbb{B}^n\left(\frac{2}{3}r\right)\right)\Big/G_p.$$

Let $\epsilon \in \mathbb{R}$ and

$$\tilde{F} = \pi_X^* F + \epsilon i\,\partial\bar{\partial}(h\iota^*\rho).$$

Then, $\tilde{F}$ is a $(1,1)$-form on $\tilde{M}$, it is positive if ϵ is small enough and satisfies $\partial\bar{\partial}\tilde{F} = 0$. It is clear that $\tilde{F} = \pi_X^* F$ on the complement of a neighborhood of E. The theorem is thus proved. □

We will obtain some applications of Theorem 5.4 in the next sections. We will show that if even we start with a nilmanifold with a big first Betti number we can get a new strong KT manifold with a considerably smaller first Betti number.

6. A simply-connected example

We are going to construct a simply-connected strong KT resolution for the quotient of the torus $\mathbb{T}^6$ by a suitable action of a finite group.

Let $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$ be the standard torus and denote by $(x_1, \dots, x_6)$ global coordinates on $\mathbb{R}^6$. Define

$$\begin{cases} \varphi^1 = dx_1 + i(f(x)\,dx_3 + dx_4), \\ \varphi^2 = dx_2 + i\,dx_5, \\ \varphi^3 = dx_3 + i\,dx_6, \end{cases} \tag{4}$$

where $f : \mathbb{R}^6 \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function. By the above expression, we easily get

$$\begin{cases} d\varphi^1 = \frac{i}{2}\,df \wedge (\varphi^3 + \bar{\varphi}^3), \\ d\varphi^j = 0, \quad j = 2, 3. \end{cases} \tag{5}$$

Taking in particular $f = f(x_3, x_6)$ we have

$$df = \frac{\partial f}{\partial x_3}\,dx_3 + \frac{\partial f}{\partial x_6}\,dx_6 = \frac{1}{2}\frac{\partial f}{\partial x_3}\big(\varphi^3 + \bar{\varphi}^3\big) + \frac{1}{2i}\frac{\partial f}{\partial x_6}\big(\varphi^3 - \bar{\varphi}^3\big).$$

Therefore, if $f = f(x_3, x_6)$ is $\mathbb{Z}^6$-periodic, then (4) defines a complex structure J on the torus $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$.

Let $\sigma : \mathbb{T}^6 \to \mathbb{T}^6$ be the involution induced by

$$(x_1, \dots, x_6) \mapsto (-x_1, \dots, -x_6).$$

By choosing $f = f(x_3, x_6)$ $\mathbb{Z}^6$-periodic and even, it follows that σ is J-holomorphic.

The set of singular points for the action of σ on $\mathbb{T}^6$ is given by

$$S = \left\{ x + \mathbb{Z}^6 \;\middle|\; x \in \frac{1}{2}\mathbb{Z}^6 \right\}$$

and consequently $(M = \mathbb{T}^6/\langle\sigma\rangle, J)$ is a complex orbifold. Note that S is a set of 64 points.

We have

$$\sigma^*(\varphi^j) = -\varphi^j, \quad j = 1, 2, 3.$$

Denote by

$$g = \frac{1}{2}\sum_{j=1}^{4}\left(\varphi^j \otimes \overline{\varphi}^j + \overline{\varphi}^j \otimes \varphi^j\right)$$

the natural σ-invariant Hermitian metric on $\mathbb{T}^6$ and by

$$F = \frac{i}{2}\sum_{j=1}^{3}\varphi^j \wedge \overline{\varphi}^j$$

the corresponding fundamental 2-form.

Proposition 6.1. $(\mathbb{T}^6/\langle\sigma\rangle, J, F)$ *is a strong (non-Kähler) KT orbifold.*

Proof. We need only to check that $\partial\overline{\partial}F = 0$. By a direct computation, taking into account (5), we get

$$\begin{aligned}
\partial\overline{\partial}F &= \frac{i}{4}\partial\left(\frac{\partial f}{\partial x_6}\varphi^3 \wedge \overline{\varphi}^3 \wedge \overline{\varphi}^1\right) \\
&= \frac{i}{8}\left[\left(\frac{\partial^2 f}{\partial x_3 \partial x_6} - i\frac{\partial^2 f}{\partial x_6^2}\right)\varphi^3\right] \wedge \varphi^3 \wedge \overline{\varphi}^3 \wedge \overline{\varphi}^1 \\
&= 0. \qquad \square
\end{aligned}$$

According to Theorem 5.4 now we may resolve the singularities of $\mathbb{T}^6/\langle\sigma\rangle$ in order to obtain a simply-connected strong KT manifold $\tilde{M}$. More precisely, for any singular point $p \in S$, we take the blow-up at p.

Applying the same argument used by Joyce as in [26, Lemmas 12.1.1 and 12.1.2], we get that the orbifold fundamental group $\pi_1(\mathbb{T}^6/\langle\sigma\rangle)$ is abelian. Hence, since

$$b_1\big(\mathbb{T}^6/\langle\sigma\rangle\big) = \dim_{\mathbb{R}}\big\{\alpha \in \Lambda^1\big(\mathbb{T}^6\big) \bigm| \alpha \text{ is harmonic and } \sigma\text{-invariant}\big\} = 0,$$

we deduce that the strong KT resolution $\tilde{M}$ of the orbifold $\mathbb{T}^6/\langle\sigma\rangle$ is simply-connected. Moreover, in a similar way we have

$$b_{2j+1}\big(\mathbb{T}^6/\langle\sigma\rangle\big) = 0 \quad \text{and} \quad b_{2j+2}\big(\mathbb{T}^6/\langle\sigma\rangle\big) = b_{2j+2}\big(\mathbb{T}^6\big),$$

for any $j = 0, 1, 2$.

Remark 6.2. The same construction can be generalized to the $2n$-dimensional orbifold $\mathbb{T}^{2n}/\langle\sigma\rangle$, where the involution $\sigma : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ is defined by

$$\sigma\big((x_1, \ldots, x_{2n})\big) = (-x_1, \ldots, -x_{2n})$$

and the complex structure on $\mathbb{T}^{2n}/\langle\sigma\rangle$ is given by

$$\begin{cases} \varphi^1 = dx_1 + i(f(x)\,dx_n + dx_{n+1}), \\ \varphi^2 = dx_2 + i\,dx_{n+2}, \\ \vdots \\ \varphi^n = dx_n + i\,dx_{2n}, \end{cases} \tag{6}$$

where $f : \mathbb{R}^{2n} \to \mathbb{R}$ is an even, $\mathbb{Z}^{2n}$-periodic $\mathcal{C}^\infty$-function of the two variables (x_n, x_{2n}).

7. 8-dimensional examples

We are going to construct two new 8-dimensional compact examples with first Betti number equal to one starting from the Kodaira–Thurston surface [27], which is the only nilmanifold of real dimension 4 admitting a strong KT structure besides the 4-dimensional torus $\mathbb{T}^4$. We will consider an action of a finite group on the product of two Kodaira–Thurston surfaces or on a product of a Kodaira–Thurston surface for $\mathbb{T}^4$ preserving the standard strong KT structures on the products. One of the two previous actions is similar to the one considered in [13] in the context of symplectic geometry.

7.1. Product of a Kodaira–Thurston surfaces and a torus

Consider on $\mathbb{C}^4$ the following product

$$\begin{aligned}&(z_1, z_2, z_3, z_4) \star (w_1, w_2, w_3, w_4)\\ &\quad = \left(z_1 + w_1, z_2 + w_2 + \frac{1}{4}i(z_1\overline{w}_1 - \bar{z}_1 w_1), z_3 + w_3, z_4 + w_4\right),\end{aligned}$$

for any $z_j, w_j \in \mathbb{C}$, $j = 1, 2, 3, 4$.

The corresponding real nilpotent Lie group N is the product $H_3 \times \mathbb{R}^5$, where H_3 is the real 3-dimensional Heisenberg group and it has a left-invariant complex J defined by the $(1,0)$-forms

$$\varphi^1 = dz_1, \qquad \varphi^2 = dz_2 - \frac{1}{4} i (z_1\, d\bar{z}_1 - \bar{z}_1\, dz_1), \qquad \varphi^3 = dz_3, \qquad \varphi^4 = dz_4.$$

Let Λ be the lattice generated by 1 and $\xi = e^{\frac{2\pi i}{3}}$ and consider the discrete subgroup $\Gamma \subset N$ formed by the elements (z_1, z_2, z_3, z_4) such that $z_1, z_2, z_3, z_4 \in \Lambda$. The compact quotient $(M = \Gamma \backslash N, J)$ is a complex nilmanifold and it can be viewed as a principal torus bundle

$$\mathbb{T}^2 = \mathbb{C}/\Lambda \to M \to \mathbb{T}^6 = (\mathbb{C}/\Lambda)^3.$$

M has a natural strong KT metric compatible with J defined by the $(1,1)$-form

$$F = \frac{i}{2} \sum_{j=1}^{4} \varphi^j \wedge \overline{\varphi}^j,$$

since

$$d\varphi^j = 0, \quad j = 1, 3, 4,$$
$$d\varphi^2 = -\frac{1}{2} i \varphi^1 \wedge \overline{\varphi}^1.$$

Consider the following action of the finite group $\mathbb{Z}_3$

$$\lambda : N \to N,$$
$$(z_1, z_2, z_3, z_4) \mapsto (\xi z_1, z_2, \xi z_3, \xi z_4).$$

One has that $\lambda(a \star b) = \lambda(a) \star \lambda(b)$, for any $a, b \in N$. Moreover, $\lambda(\Gamma) = \Gamma$ and therefore there is an induced action λ on the quotient M. Since the action on the $(1,0)$-forms is given by

$$\lambda^*\varphi^1 = \xi\varphi^1, \qquad \lambda^*\varphi^2 = \varphi^2, \qquad \lambda^*\varphi^3 = \xi\varphi^3, \qquad \lambda^*\varphi^4 = \xi\varphi^4, \tag{7}$$

the $(1,1)$-form F is $\mathbb{Z}_3$-invariant and therefore it induces a strong KT metric on the complex orbifold $\hat{M} = M/\mathbb{Z}_3$.

By Theorem 5.4 there exists a smooth compact strong KT manifold $(\tilde{M}, \tilde{F})$ which is biholomorphic to $(\hat{M}, F)$ outside the singular points.

For any singular point $p \in \hat{M}$, that we can consider as $(0,0,0,0)$ in our coordinates by translating by an element of the group N, we resolve the singularity of $\mathbb{B}^4(r)/\mathbb{Z}_3$ with a non-singular Kähler model, blowing up $\mathbb{B}^4(r)$ to get $\tilde{B}$. In this way we replace the point with a complex projective space $\mathbb{CP}^3$ in which $\mathbb{Z}_3$ acts as

$$[z_1, z_2, z_3, z_4] \to [\xi z_1, z_2, \xi z_3, \xi z_3] = \left[z_1, \xi^2 z_2, z_3, z_4\right]$$

then there are two components of the fix-point locus of the $\mathbb{Z}_3$-action on $\tilde{B}$: the point $q = [0, 1, 0, 0]$ and the complex projective plane

$$\mathbb{P}\big\{[z_1, 0, z_3, z_4]\big\} \subset \mathbb{CP}^3.$$

The resolution of $\mathbb{B}^4(r)/\mathbb{Z}_3$ is then obtained blowing-up $\tilde{B}$ at q and $\mathbb{P}\{[z_1, 0, z_3, z_4]\}$ and doing the quotient by the action of $\mathbb{Z}_3$.

Consider the $\mathbb{Z}_3$-invariant Hermitian metric on M defined by

$$g = \frac{1}{2}\sum_{j=1}^{4}\big(\varphi^j \otimes \overline{\varphi}^j + \overline{\varphi}^j \otimes \varphi^j\big).$$

By (7) we easily obtain that the only harmonic $\mathbb{Z}_3$-invariant 1-form on M is dy_2, where we denote $z_2 = x_2 + i y_2$. Since $M = \Gamma\backslash N$ is a compact nilmanifold, we have that the de Rham cohomology is just given by harmonic left-invariant forms and

$$b_k(M/\mathbb{Z}_3) = \dim_{\mathbb{R}}\big\{\alpha \in \Lambda^k(M) \;\big|\; \alpha \text{ is harmonic and } \mathbb{Z}_3\text{-invariant}\big\}.$$

Therefore, we conclude that

$$b_1(M/\mathbb{Z}_3) = 1.$$

As in [13] then one has that

$$H^1(\tilde{M}) = H^1(M/\mathbb{Z}_3) \oplus \left(\bigoplus_i H^1(E_i)\right) \tag{8}$$

where E_i is the exceptional divisor corresponding to each fixed point p_i and therefore $b_1(\tilde{M}) = b_1(M/\mathbb{Z}_3) = 1$. Therefore the topology of the new strong KT manifold is simpler with respect to the one of M since by using Nomizu's theorem [31] we had $b_1(M) = 7$ for the nilmanifold M.

7.2. *Product of two Kodaira–Thurston surfaces*

Consider on $\mathbb{C}^8$ the following product

$$\begin{aligned}&(z_1, z_2, z_3, z_4) \star (w_1, w_2, w_3, w_4)\\ &\quad = \left(z_1 + w_1, z_2 + w_2 + \frac{1}{4}i(z_1\overline{w}_1 - \bar{z}_1 w_1), z_3 + w_3, z_4 + w_4 + \frac{1}{4}i(z_3\overline{w}_3 - \bar{z}_3 w_3)\right),\end{aligned}$$

for any $z_j, w_j \in \mathbb{C}$, $j = 1, 2, 3, 4$.

The corresponding real nilpotent Lie group N is the product $H_3 \times H_3 \times \mathbb{R}^2$ and it has a left-invariant complex J defined by the $(1, 0)$-forms

$$\begin{aligned}\varphi^1 &= dz_1, &\quad \varphi^2 &= dz_2 - \frac{1}{4}i(z_1\, d\bar{z}_1 - \bar{z}_1\, dz_1),\\ \varphi^3 &= dz_3, &\quad \varphi^4 &= dz_4 - \frac{1}{4}i(z_3\, d\bar{z}_3 - \bar{z}_3\, dz_3).\end{aligned}$$

Let Λ be the lattice generated by 1 and $\xi = e^{\frac{2\pi i}{3}}$ and consider the discrete subgroup $\Gamma \subset N$ formed by the elements (z_1, z_2, z_3, z_4) such that $z_1, z_2, z_3, z_4 \in \Lambda$. The compact quotient $(M = \Gamma \backslash N, J)$ is a complex nilmanifold and it can be viewed as a principal torus bundle

$$\mathbb{T}^2 = \mathbb{C}/\Lambda \to M \to \mathbb{T}^6 = (\mathbb{C}/\Lambda)^3.$$

M has a natural strong KT metric compatible with J defined by the $(1,1)$-form

$$F = \frac{i}{2}\sum_{j=1}^{4} \varphi^j \wedge \overline{\varphi}^j,$$

since

$$\begin{aligned} d\varphi^j &= 0, \quad j = 1, 3, \\ d\varphi^2 &= -\frac{1}{2} i\varphi^1 \wedge \overline{\varphi}^1, \\ d\varphi^4 &= -\frac{1}{2} i\varphi^3 \wedge \overline{\varphi}^3. \end{aligned}$$

Consider the following action of a finite group G

$$\begin{aligned} &\lambda : N \to N, \\ &(z_1, z_2, z_3, z_4) \mapsto (\xi z_3, z_4, \xi z_1, z_2). \end{aligned}$$

One has that $\lambda(a \star b) = \lambda(a) \star \lambda(b)$, for any $a, b \in N$. Moreover, $\lambda(\Gamma) = \Gamma$ and therefore there is an induced action λ on the quotient M. Since the action on the $(1,0)$-forms is given by

$$\lambda^*\varphi^1 = \xi\varphi^3, \qquad \lambda^*\varphi^2 = \varphi^4, \qquad \lambda^*\varphi^3 = \xi\varphi^1, \qquad \lambda^*\varphi^4 = \varphi^2, \tag{9}$$

the $(1,1)$-form F is G-invariant under the previous action and therefore it induces a strong KT metric on the complex orbifold $\hat{M} = M/G$.

By Theorem 5.4 there exists a compact strong KT resolution $(\tilde{M}, \tilde{F})$ which is biholomorphic to $(\hat{M}, F)$ outside the singular set.

By using (8) and the same argument as for the previous example we get

$$b_1(M/G) = 1 = b_1(\tilde{M})$$

while for the manifold M we had $b_1(M) = 6$ by [31].

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